

PIECEWISE-KOSZUL ALGEBRAS

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ABSTRACT. It is a small step toward the Koszul-type algebras. The piecewise-Koszul algebras are, in general, a new class of quadratic algebras but not the classical Koszul ones, simultaneously they agree with both the classical Koszul and higher Koszul algebras in special cases. We give a criteria theorem for a graded algebra A to be piecewise-Koszul in terms of its Yoneda-Ext algebra $E(A)$, and show an A_∞ -structure on $E(A)$. Relations between Koszul algebras and piecewise-Koszul algebras are discussed. In particular, our results are related to the third question of Green-Marcos' [6].

1. INTRODUCTION

The Koszul algebra, introduced by S. Priddy in [14], is one of quadratic algebras with a linear resolution. Such an algebra may be understood a positively graded algebra that is “as close to semisimple as it can possible be” ([4]). Many nice homological properties of Koszul algebras have been shown in research areas of commutative and noncommutative algebras, such as algebraic topology, algebraic geometry, quantum group, and Lie algebra ([1], [4], [7], etc.). Thirty years later, motivated by the cubic Artin-Schelter regular algebras, Berger extended the concept to higher homogeneous algebras [5], one can find more discussions under the name d -Koszul algebras in [8], or higher Koszul algebras in [10], the latter explained Koszulity by A_∞ -language.

In this paper, we introduce a new class of Koszul-type algebras, we name it *piecewise-Koszul* algebra. Such an algebra is determined by a pair of parameters (p, d) , one shows its periodicity, and the other one is related to the degree of jump. It agrees with the classical Koszul algebra when the period equals to the jumping degree, and goes back to the d -Koszul algebra when the period $p = 2$.

What we are interested for these Koszul-type algebras is that in the case of $d > p \geq 3$. Such algebras provide a new class of quadratic algebras but not the classical Koszul algebras. The piecewise-Koszul algebras behave somewhat like the classical Koszul and higher Koszul algebras: for example, there is a version of the criteria theorem for a graded algebra to be Koszul in terms of its Yoneda-Ext algebra. Suppose that $A = \bigoplus_{i \geq 0} A_i$ is generated in degree zero and one with A_0 semisimple, $E(A) = \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0)$ is the Yoneda-Ext algebra of A , which is

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bigraded by the $(\text{ext}, \text{shift})$ -degree; that is, its $(i, j)^{\text{th}}$ component is $\text{Ext}_A^i(A_0, A_0)_j$. We also call $E(A)$ the Koszul dual of A , as other literatures named. The following results were stated in [7], [10] and [8], respectively:

- A is a Koszul algebra if and only if $E(A)$ is generated in the ext-degrees $0, 1$ (and $\text{Ext}_A^1(A_0, A_0) = \text{Ext}_A^1(A_0, A_0)_1$);
- A is a d -Koszul algebra if and only if $E(A)$ is generated in the ext-degrees $0, 1, 2$, and $\text{Ext}_A^2(A_0, A_0) = \text{Ext}_A^2(A_0, A_0)_d$.

For our objects, we have

- A is a piecewise-Koszul algebra if and only if $E(A)$ is generated in the ext-degrees $0, 1, p$, and $\text{Ext}_A^p(A_0, A_0) = \text{Ext}_A^p(A_0, A_0)_d$ (Theorem 2.6).

The piecewise-Koszul algebras are one of “ δ -Koszul algebras” [6]. As a result, it seems that our objects show a negative answer to the third question in [6] provided we may get a piecewise-Koszul algebra with any given period p . These are discussed in Section 2. We introduce piecewise-Koszul modules in this section as well. Denote $\mathcal{PK}(A)$ the category of piecewise-Koszul modules, we prove that $\mathcal{PK}(A)$ is extension closed and co-kernels preserved.

Although the piecewise-Koszul algebra A is quadratic in general, its Yoneda-Ext algebra $E(A)$ admits at least a non-trivial higher multiplication, which shows another difference between classical Koszul algebras and piecewise-Koszul algebras, since Keller showed in [12] that a quadratic algebra is Koszul if and only if the higher multiplications on its Koszul dual are trivial. In Section 3, we investigate the A_∞ -structure on the Koszul dual of A , where A is a piecewise-Koszul algebra. It turns out that there is a family of nontrivial higher multiplications on $E(A)$. In particular, we give a criteria for a graded algebra to be a piecewise-Koszul algebra in terms of considering the Koszul dual $E(A)$ as a $(2, d-1)$ -algebra.

However, piecewise-Koszul algebras resemble with the classical Koszul algebras on the other hand. In fact, piecewise-Koszul algebras are turned out to be closely related with Koszul algebras. We will discuss their relations in Section 4.

Let \mathbb{Z} and \mathbb{N} denote the set of integers and natural numbers respectively.

Throughout we work over a fixed field \mathbb{F} .

2. DEFINITIONS AND PROPERTIES

All the graded \mathbb{F} -algebras $A = \bigoplus_{i \geq 0} A_i$ are assumed with the following properties: (a) A_0 is a semisimple Artin algebra; (b) A is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$; (c) A_1 is of finite dimension as an \mathbb{F} -space.

The graded Jacobson radical of A , denoted by J , is $\bigoplus_{i \geq 1} A_i$. Let $Gr(A)$ denote the category of graded A -modules, and $gr(A)$ its full subcategory of finitely generated A -modules. The morphisms in these categories, denoted by $\text{Hom}_{Gr(A)}(M, N)$, are graded A -module maps of degree zero. We denote $Gr_s(A)$ and $gr_s(A)$ the

full subcategories of $Gr(A)$ and $gr(A)$ whose objects are generated in degree s , respectively. An object in $Gr_s(A)$ or $gr_s(A)$ is called a *pure* A -module.

Endowed with the Yoneda product, $\text{Ext}_A^*(A_0, A_0) = \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0)$ is an associative graded algebra. Let M and N be finitely generated graded A -modules. Then $\text{Ext}_A^*(M, N) = \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N)$ is a graded left $\text{Ext}_A^*(N, N)$ -module. For simplicity, we write $E(A) = \text{Ext}_A^*(A_0, A_0)$, and $\mathcal{E}(M) = \text{Ext}_A^*(M, A_0)$ that is a graded $E(A)$ -module, we call it the Koszul dual of M .

The Koszul dual $\mathcal{E}(M)$ of a finitely generated graded module M is bigraded; that is, if $[x] \in \text{Ext}_A^i(M, A_0)_j$, we denote the bidegree of $[x]$ as (i, j) , call the first degree *ext*-degree and the second degree *shift*-degree, respectively. Similarly, the Yoneda-Ext-algebra $E(A)$ of a positively graded \mathbb{F} -algebra is a bigraded algebra.

Given a pair of integers d and p ($d \geq p \geq 2$), we introduce a function $\delta_p^d : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\delta_p^d(n) = \begin{cases} \frac{nd}{p}, & \text{if } n \equiv 0(\text{mod } p), \\ \frac{(n-1)d}{p} + 1, & \text{if } n \equiv 1(\text{mod } p), \\ \dots & \dots \\ \frac{(n-p+1)d}{p} + p-1, & \text{if } n \equiv p-1(\text{mod } p). \end{cases}$$

Definition 2.1. A graded algebra $A = \bigoplus_{i \geq 0} A_i$ is called a *piecewise-Koszul algebra* if the trivial A -module A_0 admits a minimal graded projective resolution

$$\mathbf{P} : \dots \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow A_0 \rightarrow 0,$$

such that each P_n is generated in degree $\delta_p^d(n)$ for all $n \geq 0$.

The parameter p shows certain periodicity in the resolution above, and the parameter d is related to a gap between two segments. The classical Koszul algebras and the higher Koszul algebras are piecewise-Koszul algebras, which take place of $d = p$ and $p = 2$ respectively, we refer to [4]-[14] for the details. What we are interested, in this paper, is the case of $d > p \geq 3$. Such algebras provide us a new class of quadratic algebras different from the classical Koszul algebras.

We will give an example of piecewise-Koszul algebra at the end of this section.

Definition 2.2. Let A be a piecewise-Koszul algebra and $M \in gr(A)$. We call M a *piecewise-Koszul module* if it has a minimal graded projective resolution of the form

$$\mathbf{Q} : \dots \rightarrow Q_n \xrightarrow{f_n} \dots \rightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \rightarrow 0,$$

in which each Q_n is generated in degree $\delta_p^d(n) + s$ for $n \geq 0$, where s is a fixed integer.

It is clear, by the definitions above, that a graded \mathbb{F} -algebra A is a piecewise-Koszul algebra (with the period p and jump degree d) if and only if the n^{th} component $\text{Ext}_A^n(A_0, A_0)$ is concentrated in degree $(n, \delta_p^d(n))$ for all $n \geq 0$. Let A be a piecewise-Koszul algebra and $M \in gr(A)$, then M is a piecewise-Koszul module if and only if $\text{Ext}_A^n(M, A_0)$ is concentrated in the degree $(n, \delta_p^d(n) + s)$ for all $n \geq 0$, where d, p, s are fixed integers in the definition.

Definition 2.3. [13] An A_∞ -algebra over a field \mathbb{F} is a \mathbb{Z} -graded vector space

$$E = \bigoplus_{n \in \mathbb{Z}} E^n$$

endowed with a family of graded \mathbb{F} -linear maps

$$m_n : E^{\otimes n} \longrightarrow E, \quad (n \geq 1)$$

of degree $(2 - n)$ satisfying the following *Stasheff's identities*: for all $n \geq 1$,

$$\mathbf{SI}(n) \quad \sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0,$$

where the sum runs over all decomposition $n = r + s + t$, $r, t \geq 0$ and $s \geq 1$, and where $u = r + 1 + t$. Note that when these formulas are applied to homogeneous elements, additional signs appear due to the Koszul sign rule. If E is bi-graded, we can also define bi-graded A_∞ -algebras similarly and all the multiplications preserve the second degree, we refer to [13] for the details.

The multiplications $\{m_n\}_{n \geq 3}$ are called the higher multiplications of E .

An *augmented A_∞ -algebra* E is an A_∞ -algebra satisfying the following conditions, where for the notions of *A_∞ -morphism* and *strict A_∞ -morphism*, we refer to [10] and [13] for the details,

- (1) There is a strict A_∞ -morphism $\eta_E : \mathbb{F} \rightarrow E$ such that $m_n(1^{\otimes i} \otimes \eta_E \otimes 1^{\otimes j}) = 0$ for all $n \neq 2$ and $i + j = n - 1$, and $m_2(1 \otimes \eta_E) = m_2(\eta_E \otimes 1) = 1_E$;
- (2) There is a strict A_∞ -morphism $\varepsilon_E : E \rightarrow \mathbb{F}$ such that $\varepsilon_E \circ \eta_E = 1$.

The following definition is related to the Koszul dual $E(A)$ as a special A_∞ -algebras, where A is a piecewise-Koszul algebra.

Definition 2.4. [10] Let $E = \bigoplus_{n \in \mathbb{Z}} E^n$ be an A_∞ -algebra. If E has only two nontrivial multiplications m_2 and m_l , then (E, m_2, m_l) is called a $(2, l)$ -algebra.

An augmented $(2, l)$ -algebra (E, m_2, m_l) is called a *reduced $(2, l)$ -algebra* provided the following conditions are satisfied:

- (1) $E = \mathbb{F} \oplus E^1 \oplus E^2 \oplus \dots$;
- (2) $m_2(E^{3k_1+s_1} \otimes E^{3k_2+s_2}) = 0$ for all $3 \leq s_1 + s_2 \leq 4$ and $s_j = 1$ or 2 and $k_j \geq 0$, $j = 1, 2$;
- (3) $m_l(E^{i_1} \otimes E^{i_2} \otimes \dots \otimes E^{i_l}) = 0$ unless one of the $i_j \equiv 2 \pmod{3}$ and the others $\equiv 1 \pmod{3}$.

A reduced $(2, l)$ -algebra E is said to be generated by E^1 if for all $n \geq 2$,

$$E^n = \sum_{i+j=n; i, j \geq 1} m_2(E^i \otimes E^j) + \sum m_l(E^{i_1} \otimes E^{i_2} \otimes \dots \otimes E^{i_l}),$$

where the sum in the second sigma runs over all decompositions $i_1 + i_2 + \dots + i_l + 2 - l = n$ and all $i_t \geq 1$ and $t = 1, 2, \dots, l$.

In what follows, we use $\Omega^n(M)$ to denote the n^{th} syzygy of M .

We have the following equivalent descriptions on piecewise-Koszul objects and Theorem 2.6 seems to give a negative answer to Green's open question in [6].

Lemma 2.5. [8] Suppose that \mathbf{P} is the minimal graded projective resolution of the trivial A -module A_0 and P_n is finitely generated with generators in degree $\delta_p^d(n)$. Assume that $\delta_p^d(i+j) = \delta_p^d(i) + \delta_p^d(j)$. Then the Yoneda map

$$\mathrm{Ext}_A^i(A_0, A_0) \otimes_{\mathbb{F}} \mathrm{Ext}_A^j(A_0, A_0) \rightarrow \mathrm{Ext}_A^{i+j}(A_0, A_0)$$

is surjective. Moreover,

$$\mathrm{Ext}_A^{i+j}(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0) \cdot \mathrm{Ext}_A^j(A_0, A_0) = \mathrm{Ext}_A^j(A_0, A_0) \cdot \mathrm{Ext}_A^i(A_0, A_0).$$

□

Theorem 2.6. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded \mathbb{F} -algebra as assumed before, and $E(A)$ its Yoneda-Ext-algebra. Then the following statements are equivalent:

- (1) A is a piecewise-Koszul algebra;
- (2) $E(A)$ is generated in the ext-degrees $0, 1$ and p , moreover, $\mathrm{Ext}_A^p(A_0, A_0) = \mathrm{Ext}_A^p(A_0, A_0)_d$.

Proof. We write the function $\delta_p^d(n)$ as $\delta(n)$ for simplicity.

(1) \Rightarrow (2). This follows from Lemma 2.5 and an easy induction on the ext-degrees.

(2) \Rightarrow (1). Suppose that the minimal projective resolution of A_0 is

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A_0 \rightarrow 0.$$

When $0 \leq n < p$, we have $\mathrm{Ext}_A^n(A_0, A_0) = \mathrm{Ext}_A^n(A_0, A_0)_n = (\mathrm{Ext}_A^1(A_0, A_0))^n$.

Hence P_n is generated in degree $\delta(n)$ for $0 \leq n < p$.

In general, we shall show that P_n is generated in degree $\delta(n)$ for all $n \geq 0$.

We only prove the case of $k = i = 1$. Observing that $\mathrm{Ext}_A^p(A_0, A_0) = \mathrm{Ext}_A^p(A_0, A_0)_d$ and $\mathrm{Ext}_A^s(A_0, A_0) \cdot \mathrm{Ext}_A^t(A_0, A_0) = 0$ for $s, t < p$, we have

$$\begin{aligned} & \mathrm{Ext}_A^{p+1}(A_0, A_0) \\ &= \sum_{s+t=p+1; s, t \leq p} \mathrm{Ext}_A^s(A_0, A_0) \cdot \mathrm{Ext}_A^t(A_0, A_0) \\ &= \sum_{s+t=p+1, s, t < p} \mathrm{Ext}_A^s(A_0, A_0) \cdot \mathrm{Ext}_A^t(A_0, A_0) \\ & \quad + \mathrm{Ext}_A^1(A_0, A_0) \cdot \mathrm{Ext}_A^p(A_0, A_0) + \mathrm{Ext}_A^p(A_0, A_0) \cdot \mathrm{Ext}_A^1(A_0, A_0) \\ &= \mathrm{Ext}_A^1(A_0, A_0) \cdot \mathrm{Ext}_A^p(A_0, A_0) + \mathrm{Ext}_A^p(A_0, A_0) \cdot \mathrm{Ext}_A^1(A_0, A_0) \\ &= \mathrm{Ext}_A^1(A_0, A_0)_1 \cdot \mathrm{Ext}_A^p(A_0, A_0)_d + \mathrm{Ext}_A^p(A_0, A_0)_d \cdot \mathrm{Ext}_A^1(A_0, A_0)_1 \\ &= \mathrm{Ext}_A^{p+1}(A_0, A_0)_{d+1}. \end{aligned}$$

Hence P_{p+1} is generated in degree $\delta(p+1)$.

□

There is a similar description for piecewise-Koszul modules.

Theorem 2.7. Let A be a piecewise-Koszul algebra and $M \in \mathrm{gr}_0(A)$. Then M is a piecewise-Koszul module if and only if $\mathcal{E}(M)$ is generated in degree 0 as a graded $E(A)$ -module.

Proof. This is an easy consequence of Proposition 3.5 in [8].

□

Let A be a piecewise-Koszul algebra and $M \in gr_s(A)$ be a piecewise-Koszul module. Then we have the following exact sequences

$$0 \rightarrow \Omega^n(M) \rightarrow \Omega^n(M/JM) \rightarrow \Omega^{n-1}(JM) \rightarrow 0,$$

for all $n \geq 1$. All modules in the exact sequences above are generated in degree $\delta_p^d(n) + s$. We get the following exact sequences,

$$0 \rightarrow \text{Ext}_A^{n-1}(JM, A_0) \rightarrow \text{Ext}_A^n(M/JM, A_0) \rightarrow \text{Ext}_A^n(M, A_0) \rightarrow 0.$$

Moreover, in the shift-grading, all the modules in the exact sequences above are concentrated in degree $\delta_p^d(n) + s$ for a fixed integer s .

The following proposition shows that the category $\mathcal{PK}(A)$ of piecewise-Koszul modules is closed under extension and preserves cokernels. We omit the proofs since they are obvious.

Proposition 2.8. *Let*

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$$

be a short exact sequence in $gr(A)$. Then we have the following statements,

- (1) *If K and N are in $\mathcal{PK}(A)$ with K and N being in $gr_s(A)$, then $M \in \mathcal{PK}(A)$,*
- (2) *If K and M are in $\mathcal{PK}(A)$ with K and M being in $gr_s(A)$, then $M/K \cong N \in \mathcal{PK}(A)$.*

□

Proposition 2.9. *Let $M \in gr(A)$ be a piecewise-Koszul module. Then*

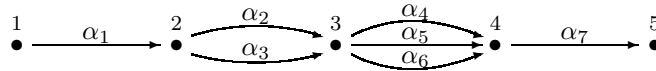
- (1) *All the $(pn)^{th}$ syzygies $\Omega^{pn}(M)$ are piecewise-Koszul modules;*
- (2) *All the $(pn-1)^{th}$ syzygies $\Omega^{pn-1}(JM)$ are piecewise-Koszul modules.*

Proof. The first assertion is immediate from the definition of the piecewise-Koszul modules. For the second statement, clearly, we have the following exact sequences in $gr_{\delta_p^d(pn)+s}(A)$

$$0 \rightarrow \Omega^{pn}(M) \rightarrow \Omega^{pn}(M/JM) \rightarrow \Omega^{pn-1}(JM) \rightarrow 0,$$

by assertion (1), $\Omega^{pn}(M)$ and $\Omega^{pn}(M/JM)$ are in $\mathcal{PK}(A)$ since M and M/JM are in $\mathcal{PK}(A)$. By Proposition 2.8, we have that $\Omega^{pn-1}(JM) \in \mathcal{PK}(A)$. □

Example 2.10. Let \mathbb{F} be a field and let Γ be the quiver:



Now let $A = \mathbb{F}\Gamma/R$, where R is the ideal generated by the following relations:

$$\alpha_1\alpha_2 - \alpha_1\alpha_3, \quad \alpha_4\alpha_7 - \alpha_5\alpha_7, \quad \alpha_5\alpha_7 - \alpha_6\alpha_7, \quad \alpha_2\alpha_4, \quad \alpha_3\alpha_6.$$

It is not difficult to check that A is a piecewise-Koszul algebra with $p = 3$ and $d = 4$.

3. THE A_∞ -STRUCTURE ON THE KOSZUL DUAL $E(A)$

Let A be a piecewise-Koszul algebra and $E(A)$ its Koszul dual. We will investigate the A_∞ -structure on $E(A)$ in detail in this section. As a result, a criteria for a graded algebra to be piecewise-Koszul in terms of the A_∞ -algebra is given.

For the simplicity, we assume, in this section, that $A_0 = \mathbb{F}$ in the graded algebra A and the piecewise-Koszul objects are related to $\delta_3^d(n)$ with $d > 3$ a fixed integer.

Lemma 3.1. ([12], [10]) *Let A be an augmented differential bi-graded algebra and $E = HA$. Then there is an augmented A_∞ -structure $\{m_i\}$ on E such that $m_1 = 0$, m_2 is induced by the multiplication of A , and E is quasi-isomorphic to A as augmented bi-graded A_∞ -algebras.*

Let $A = \mathbb{F} \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded algebra with $\dim_{\mathbb{F}}(A_i) < \infty$ for all i . Let J be the graded Jacobson radical of A , J^* be the graded dual of J and $T(J^*)$ the tensor algebra. From ([12], [10]), we know that $T(J^*)$ is an augmented differential bi-graded algebra and $E(A) = \text{Ext}_A^*(\mathbb{F}, \mathbb{F}) \cong H(T(J^*))$. By Lemma 3.1, the Koszul dual of A , $E(A)$, is an augmented bi-graded A_∞ -algebra up to quasi-isomorphism. We will call such A_∞ -structure on $E(A)$ induced from $T(J^*)$.

Proposition 3.2. *Let A be a piecewise-Koszul algebra and $E(A)$ be its Koszul dual. Then $\text{Ext}_A^3(\mathbb{F}, \mathbb{F}) = \text{Ext}_A^3(\mathbb{F}, \mathbb{F})_d$, and all possible A_∞ -structures $\{m_q\}$ on $E(A)$ satisfy $m_p = 0$ if $q \neq k(d-3) + 2$ for all $k \in \mathbb{N}$. Moreover, $(E(A), \{m_q\})$ satisfies the conditions (1) and (2) in the Definition 2.4.*

Proof. Since A is a piecewise-Koszul algebra, we have $E^{3k+s} := \text{Ext}_A^{3k+s}(\mathbb{F}, \mathbb{F}) = E_{kd+s}^{3k+s}$ where $s = 0, 1$ or 2 . Recall that all the multiplications $\{m_q\}$ preserve the second degree, we have

$$m_q(E^{3k_1+s_1} \otimes E^{3k_2+s_2} \otimes \cdots \otimes E^{3k_q+s_q}) \subset E_{(k_1+k_2+\cdots+k_q)d+s_1+s_2+\cdots+s_q}^{3(k_1+k_2+\cdots+k_q)+s_1+s_2+\cdots+s_q+2-q},$$

where $s_i = 0, 1$ or 2 ($1 \leq i \leq q$). When $s_1 + s_2 + \cdots + s_q + 2 - q = 3k$ and if $m_q \neq 0$, then $s_1 + s_2 + \cdots + s_q = kd$ and $q = k(d-3) + 2$. For the cases $s_1 + s_2 + \cdots + s_q + 2 - q = 3k + 1$ and $s_1 + s_2 + \cdots + s_q + 2 - q = 3k + 2$, similarly, we can get the same result. For the last statement, it is clear that $E^0 = \mathbb{F}$. Notice that $d > 3$, we have

$$m_2(E_{k_1d+1}^{3k_1+1} \otimes E_{k_2d+2}^{3k_2+2}) \subset E_{(k_1+k_2)d+3}^{3(k_1+k_2)+3} = 0,$$

$$m_2(E_{k_1d+2}^{3k_1+2} \otimes E_{k_2d+1}^{3k_2+1}) \subset E_{(k_1+k_2)d+3}^{3(k_1+k_2)+3} = 0,$$

and

$$m_2(E_{k_1d+2}^{3k_1+2} \otimes E_{k_2d+2}^{3k_2+2}) \subset E_{(k_1+k_2)d+4}^{3(k_1+k_2)+4} = 0.$$

□

If E is an A_∞ -algebra with nontrivial multiplications m_2 and m_l , then it is easy to see that all the Stasheff identities hold automatically except for the following three cases:

SI(3):
$$m_2(m_2 \otimes 1) = m_2(1 \otimes m_2),$$

$$\mathbf{SI}(2l-1): \quad \sum_{i+j=l-1} (-1)^{i+l} m_l(1^{\otimes i} \otimes m_l \otimes 1^{\otimes j}) = 0,$$

and

$$\mathbf{SI}(l+1): \quad \sum_{i+j=l-1} (-1)^{i+l} m_l(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) = m_2(1 \otimes m_l) - (-1)^l m_2(m_l \otimes 1).$$

From [11], we have the following lemma.

Lemma 3.3. *Let $A = \mathbb{F} \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded algebra generated in degree 1. Then there exists an A_∞ -structure on $\text{Ext}_A^*(\mathbb{F}, \mathbb{F})$, such that $\text{Ext}_A^*(\mathbb{F}, \mathbb{F})$ is generated by $\text{Ext}_A^1(\mathbb{F}, \mathbb{F})$ as an A_∞ -algebra.*

Remark 3.1. By Lemma 3.3, it is easy to see that there exists at least one nontrivial higher multiplication constructed in Proposition 3.2.

Theorem 3.4. *Let $A = \mathbb{F} \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded algebra generated in degree 1 and $E(A)$ be its Koszul dual.*

- (1) *If A is a piecewise-Koszul algebra, then there exists an A_∞ -structure $\{m_q\}$ (the same as Proposition 3.2) such that $(E(A), \{m_q\})$ is an A_∞ -algebra generated by $E^1(A)$;*
- (2) *Conversely, if the nontrivial multiplications in the A_∞ -structure on $E(A)$ induced from $T(J^*)$ are m_2 and m_{d-1} , such that $(E(A), m_2, m_{d-1})$ is a reduced $(2, d-1)$ -algebra generated by $E^1(A)$. Then A is a piecewise-Koszul algebra with jumping degree d .*

Proof. The statement (1) follows from Proposition 3.2 and Lemma 3.3. Since A is generated by A_1 , we have $E^1(A) = E^1(A)_1$. Therefore $E^2(A) = E^2(A)_2$, since $E^2(A) = m_2(E^1(A) \otimes E^1(A))$. Also

$$E^3(A) = \sum_{i_1+i_2+\cdots+i_{d-1}+2-(d-1)=3, i_s \geq 1} m_{d-1}(E^{i_1}(A) \otimes E^{i_2}(A) \otimes \cdots \otimes E^{i_{d-1}}(A))$$

implies that $E^3(A) = E^3(A)_{\delta_3^d(3)}$. Now assume that we have the identities: $E^n(A) = E^n(A)_{\delta_3^d(n)}$ for all $n < 3k$, where $k \in \mathbb{N}$. Now consider $E^{3k}(A)$, $E^{3k+1}(A)$ and $E^{3k+2}(A)$ respectively. By hypothesis,

$$\begin{aligned} E^{3k}(A) &= \sum_{i+j=3k; i, j \geq 1} m_2(E^i(A) \otimes E^j(A)) \\ &+ \sum_{i_1+i_2+\cdots+i_{d-1}+2-(d-1)=3k, i_s \geq 1} m_{d-1}(E^{i_1}(A) \otimes E^{i_2}(A) \otimes \cdots \otimes E^{i_{d-1}}(A)) \\ &= \sum_{3p+3q=3k; p, q \geq 1} m_2(E^{3p}(A) \otimes E^{3q}(A)) \\ &+ \sum_{i_1+i_2+\cdots+i_{d-1}+2-(d-1)=3k, i_s \geq 1} m_{d-1}(E^{i_1}(A) \otimes E^{i_2}(A) \otimes \cdots \otimes E^{i_{d-1}}(A)), \\ &= E^{3k}(A)_{\delta_3^d(3k)}. \end{aligned}$$

$$\begin{aligned}
E^{3k+1}(A) &= \sum_{i+j=3k+1; i,j \geq 1} m_2(E^i(A) \otimes E^j(A)) \\
&+ \sum_{i_1+i_2+\dots+i_{d-1}+2-(d-1)=3k+1, i_s \geq 1} m_{d-1}(E^{i_1}(A) \otimes E^{i_2}(A) \otimes \dots \otimes E^{i_{d-1}}(A)) \\
&= \sum_{p+q=k; p, q \geq 1} m_2(E^{3p+1}(A) \otimes E^{3q}(A) + E^{3p}(A) \otimes E^{3q+1}(A)) \\
&= E^{3k+1}(A)_{\delta_3^d(3k+1)}.
\end{aligned}$$

And similarly, we can prove $E^{3k+2}(A) = E^{3k+2}(A)_{\delta_3^d(3k+2)}$. Therefore, A is a piecewise-Koszul algebra with jumping degree d . \square

4. KOSZUL OBJECTS FROM PIECEWISE-KOSZUL OBJECTS

We will construct Koszul objects from the given piecewise-Koszul objects in this section.

Let A be a piecewise-Koszul algebra and $M \in gr_0(A)$ be a piecewise-Koszul module respect to $\delta_3^d(n)$. For $k \geq 1$, we set

$$\mathbf{E}_k(A) = \bigoplus_{n \geq 0} \text{Ext}_A^{3kn}(A_0, A_0)$$

and

$$\mathbf{E}_k(M) = \bigoplus_{n \geq 0} \text{Ext}_A^{3kn}(M, A_0).$$

Naturally $\mathbf{E}_k(A)$ is a subalgebra of $E(A)$, and $\mathbf{E}_k(M)$ is a graded $\mathbf{E}_k(A)$ -module.

Theorem 4.1. *Let A be a piecewise-Koszul algebra and M be a piecewise-Koszul module respect to $\delta_3^d(n)$. The notations $\mathbf{E}_k(A)$ and $\mathbf{E}_k(M)$ are as defined above. Then for all integers $k \geq 1$, we have*

- (1) $\mathbf{E}_k(M) = \bigoplus_{n \geq 0} \text{Ext}_A^{3kn}(M, A_0)$ is a Koszul $\mathbf{E}_k(A)$ -module, and
- (2) $\mathbf{E}_k(A) = \bigoplus_{n \geq 0} \text{Ext}_A^{3kn}(A_0, A_0)$ is a Koszul algebra.

Proof. By Theorem 2.7, we get that $\mathbf{E}_k(M)$ is generated in degree 0 as a graded $\mathbf{E}_k(A)$ -module.

Clearly, we have the following exact sequences, for all $n, k > 0$,

$$0 \rightarrow \text{Ext}_A^{3kn-1}(JM, A_0) \rightarrow \text{Ext}_A^{3kn}(M/JM, A_0) \rightarrow \text{Ext}_A^{3kn}(M, A_0) \rightarrow 0$$

such that all the modules in the above exact sequences are concentrated in degree $\delta_3^d(3nk)$ in the shift-grading.

We have the following exact sequences

$$0 \rightarrow \text{Ext}_A^{3k(n-1)}(\Omega^{3k-1}(JM), A_0) \rightarrow \text{Ext}_A^{3kn}(M/JM, A_0) \rightarrow \text{Ext}_A^{3kn}(M, A_0) \rightarrow 0$$

since

$$\text{Ext}_A^{3kn-1}(JM, A_0) = \text{Ext}_A^{3k(n-1)}(\Omega^{3k-1}(JM), A_0).$$

By taking the direct sums of the above exact sequences, we have

$$0 \rightarrow \mathbf{E}_k(\Omega^{3k-1}(JM))[1] \rightarrow \bigoplus_{n > 0} \text{Ext}_A^{3kn}(M/JM, A_0) \rightarrow \bigoplus_{n > 0} \text{Ext}_A^{3kn}(M, A_0) \rightarrow 0.$$

Now we claim that $\mathbf{E}_{\mathbf{k}}(M/JM)$ is a projective cover of $\mathbf{E}_{\mathbf{k}}(M)$ and it is generated in degree 0. In fact, $\mathbf{E}_{\mathbf{k}}(M/JM)$ is a $\mathbf{E}_{\mathbf{k}}(A)$ -projective module since M/JM is semi-simple. M/JM is a piecewise-Koszul module since A is a piecewise-Koszul algebra. Hence $\mathbf{E}_{\mathbf{k}}(M/JM)$ is generated in degree 0 as a graded $\mathbf{E}_{\mathbf{k}}(A)$ -module, and by the above exact sequence, it is the graded projective cover of $\mathbf{E}_{\mathbf{k}}(M)$.

Therefore the first syzygy is $\bigoplus_{n>0} \text{Ext}_A^{3k(n-1)}(\Omega^{3k-1}(JM), A_0)$, and $\Omega^{3k-1}(JM)$ is generated in degree $\delta_3^d(3k)$ and by Proposition 2.9, $\Omega^{3k-1}(JM)$ is again a piecewise-Koszul module. Inductively, we prove the first assertion.

The second assertion is the case of $M = A_0$. The proof is finished. \square

Remark 4.1. If the piecewise-Koszul objects are respect to $\delta_p^d(n) + s$ in the above theorem, then the theorem can be restated as follows: Let A be a piecewise-Koszul algebra and M be a piecewise-Koszul module. Then for all integers $k \geq 1$, we have

- (1) $\mathcal{E}_{\mathbf{k}}(M) = \bigoplus_{n \geq 0} \text{Ext}_A^{pkn}(M, A_0)$ is a Koszul module, and
- (2) $\mathcal{E}_{\mathbf{k}}(A) = \bigoplus_{n \geq 0} \text{Ext}_A^{pkn}(A_0, A_0)$ is a Koszul algebra.

By Theorem 4.1, we can construct new Koszul objects from the given piecewise-Koszul objects. However, we can't construct d -Koszul objects from a given piecewise-Koszul object for $d > 2$.

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REFERENCES

1. R. M. Aquino and E. L. Green, *On modules with linear presentations over Koszul algebras*, Comm. Algebra **33** (2005) 19-36.
2. M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge studies in Advanced Mathethematics, Vol. **36**, Cambridge. UK: Cambridge University Press (1995).
3. M. Artin and W. F. Schelter, *Graded algebras of global dimension 3*, Adv. math., Vol. **66** (1987), 171-216.
4. A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., Vol. **9** (1996), 473-525.
5. R. Berger, *Koszulity for nonquadratic algebras*, J. Alg., Vol. **239** (2001), 705-734.
6. E. L. Green, E. N. Marcos, *δ -Koszul algebras*, Comm. Alg., Vol. **33**(6) (2005), 1753-1764.
7. E. L. Green, R. Martinez-Villa, *Koszul and Yoneda algebras*, Representation theory of algebras (Cocoyoc, 1994), CMS Conference Proceedings, Vol. **18**, American Mathematical Society, Providence, RI, (1996), 247-297.
8. E. L. Green, E. N. Marcos, R. Martinez-Villa, Pu Zhang, *D-Koszul algebras*, J. pure and Appl. Algebra **193** (2004), 141-162.
9. E. L. Green, R. Martinez-Villa, I. Reiten, ϕ . Solberg, D. Zacharia, *On modules with linear presentations*, J. Alg., **205**(2) (1998), 578-604.
10. J.-W. He, D.-M. Lu, *Higher Koszul Algebras and A-infinity Algebras*, J. Alg., **293** (2005), 335-362.
11. B. Keller, *A-infinity algebras in representation theory*, Contribution to the proceedings of ICRA IX, Beijing 2000.

12. B. Keller, *Introduction to A-infinity algebras and modules*, Homology Homotopy appl., **3** (2001), (electronic), 1-35.
13. D.-M Lu, J. H. Palmieri, Q.-s. Wu and J. J. Zhang, *A_∞ -algebras for ring theorists*, Alg. Colloq., Vol. 11(1), 91-128, 2004.
14. S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc., Vol. **152** (1970), 39-60.
15. C. A. Weible, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, Vol. **38**, Cambridge Univ. Press, (1995).

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